The Solow Growth Model

Reading:

Romer, Chapter 1;
Robert E. Lucas Jr., "Why Doesn't Capital Flow from Rich to Poor Countries?" AER Papers and Proceedings, 92-96, 1990;
Martin Feldstein and Charles Horioka, "Domestic Savings and International Capital Flows," Economic Journal, 314-329, 1980.
Mankiw, Romer and Weil. "A Contribution to the Empirics of Economic Growth," Quarterly Journal of Economics 107 (2), 1992, 407-437.

Stylized Facts of Economic Growth

Little growth in per-capita output before the Industrial Revolution: technological progress and productivity gains translated into population growth rather than output growth; most people attained little more than subsistence income throughout most of human history.

Average growth in GDP per capita: 0% in Western Europe and India during the first millenium, 0.14% in Western Europe and 0.02% in India between 1000 and 1820. Population growth was similarly 0% during the first millenium and 0.2% in Western Europe and 0.13% in India between 1000 and 1820 [Maddison, 2001].

World population grew on average less than 0.1% per year between 1 and 1750 [Livi-Baci, *A Concise History of World Population*, Blackwell, 1997].

Rapid growth in output and standard of living since the Industrial Revolution: per-capita incomes increased between 50 and 300 times in Western Europe and the US during the past 200 years.

The Great Divergence: little difference in per capita incomes across countries until the Industrial Revolution, on-going divergence since then. The ratio between per capita incomes in the richest regions (US, Canada and New Zealand) and the poorest ones (Africa) was 3 in 1820; this ratio has increased to 20 by 1998 [Maddison, 2001].

By the end of the first millenium, China and India were both richer and more technologically advanced then Europe. By 1700, the core areas of Europe and Asia had similar consumption levels and overall wellbeing. Subsequently, Industrial Revolution took place in Europe whose growth has far outpaced that of China and India.

Different explanations for divergence and reversal or roles:

- Landes (*The Wealth and Poverty of Nations: Why Some Are So Rich and Some Are So Poor*, 1999): European nations embraced *insitutions* that encouraged entrepreneurship, invention and technological progress. Competition between nations created a strong incentive to utilize technological progress to gain a competitive (and military) advantage. The latter was not the case in China, India or the Ottoman empire which were subject to political hegemony and (initially) free from an outside threat.

- Pomeranz (*The Great Divergence: China, Europe, and the Making of the Modern World Economy*, 2001): Europe benefited from having convenient sources of coal that replaced timber at the eve of the Industrial Revolution. Because of trade and colonization of the New World, Europe gained a new source of primary products. This allowed increased population growth in Europe and, by releasing labor from aggriculture, facilitated its specialiazation in manufacturing .

Growth often varies considerably over time: the same country may experience growth accelerations and slowdowns: Mexico, Soviet Union/Russia, Japan.

Growth miracles – South Korea, Taiwan, Singapore, Hong Kong, and more recently China and Botswana – and growth disasters – Argentina (one of the richest countries in 1900) and Sub-saharan Africa.

Cross-country differences in growth rates and output levels are correlated with differences in other measures of well-being: nutrition, literacy, infant mortality, life expectancy, etc.

Lucas (1988): *Once one starts to think about economic growth, it is hard to think about anything else.*

Potential sources of growth:

- factor accumulation: physical capital, labor and human capital;
- technological progress;
- institutions.

Neoclassical economics initially focused on factor accumulation (Solow and Ramsey models of growth), then on technological progress (endogenous growth models). More recently, institutional economics emphasises the importance of institutions.

The Solow Growth Model

Robert Solow (1956), T.W. Swan (1956).

Assumptions

Savings and investment decisions are exogenous (no individual optimization). Factor accumulation and technological growth are also exogenous.

Production function, with physical capital K, labor L and knowledge or technology A:

$$Y(t) = F(K(t), A(t)L(t))$$

Time affects output only through K, L and A. Technology is *labor-augmenting*: AL is *effective labor*. Land and natural resources are ignored (not considered among factors of production).

Constant returns to scale (CRTS): $F(cK, cAL) = cF(K, AL)$ for any $c \geq 0$.

Example: Cobb-Douglas function

$$Y = K^{\alpha}(AL)^{1-\alpha}$$

displays CRTS:

$$\begin{aligned} & (cK)^{\alpha}(cAL)^{1-\alpha} \\ = \; & c^{\alpha}K^{\alpha}c^{1-\alpha}(AL)^{1-\alpha} \\ = \; & cK^{\alpha}(AL)^{1-\alpha} \end{aligned}$$

Because of CRTS, we can express $Y(t)$ in *intensive form*, taking $c = \frac{1}{AL}$:

$$F\left(\frac{K}{AL}, 1\right) = \frac{1}{AL}F(K, AL)$$

or, denoting $k = \frac{K}{AL}$, $y = \frac{Y}{AL}$ and $f(k) = F(k, 1)$, we can relate output per unit of effective labor as a function of capital per unit of effective labor:

$$y = f(k).$$

The intensive-form production function is assumed to have the following properties:

$$f(0) = 0$$

$$f\prime(k) > 0$$
$$f''(k) < 0$$

and Inada conditions:

$$\lim_{k\to 0} f\prime(k) = \infty$$
$$\lim_{k\to\infty} f\prime(k) = 0$$

Example: Cobb-Douglas function:

$$Y = K^{\alpha} (AL)^{1-\alpha}$$

which translates into

$$y = k^{\alpha}.$$

Diminishing marginal product of capital:

$$\begin{aligned} f\prime(k) &= \alpha k^{\alpha-1} > 0 \\ f"(k) &= -\alpha(1-\alpha) k^{\alpha-2} < 0 \end{aligned}$$

Note that $f\prime(k) = \frac{\partial Y}{\partial K}$:

$$\begin{aligned} \frac{\partial Y}{\partial K} &= \alpha K^{\alpha-1} (AL)^{1-\alpha} \\ &= \alpha \frac{K^{\alpha-1}}{(AL)^{\alpha-1}} \\ &= \alpha k^{\alpha-1} \end{aligned}$$

Evolution of Effective Labor

Labor and knowledge grow at constant exogenous rates n and g:

$$\dot{L}(t) = nL(t)$$

$$\dot{A}(t) = gA(t)$$

Note that the growth rate of a variable equals the rate of change of its log:

$$\frac{d\ln L(t)}{dt} = \frac{d\ln L(t)}{dL(t)}\frac{dL(t)}{dt} = \frac{1}{L(t)}\dot{L}(t) = \frac{\dot{L}(t)}{L(t)} = n$$

Using this, and taking the initial value of $L(t)$ as given:

$$\begin{aligned} \frac{d\ln L(t)}{dt} &= n \\ d\ln L(t) &= ndt \\ \ln L(1) - \ln L(0) &= n \end{aligned}$$

Therefore,

$$\begin{aligned} \ln L(1) &= \ln L(0) + n \\ \ln L(2) &= \ln L(1) + n = \ln L(0) + 2n \\ &\ldots \\ \ln L(t) &= \ln L(0) + nt \end{aligned}$$

Similarly,

$$\ln A(t) = \ln A(0) + gt$$

Hence, for given initial levels of L and A, this implies that labor and knowledge grow exponentially:

$$L(t) = L(0)\, e^{nt}$$

$$A(t) = A(0)\, e^{gt}$$

Dynamics of Capital

Assume constant and exogenous savings rate, s, (i.e. not a result of individual optimization decision) and constant depreciation rate of capital, δ:

$$\dot{K}(t) = sY(t) - \delta K(t)$$

Dynamics of capital per unit of effective labor, $k = \frac{K}{AL}$:

$$\begin{aligned}
\dot{k}(t) &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{[A(t)L(t)]^2}\left[A(t)\dot{L}(t) + \dot{A}(t)L(t)\right] \\
&= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)}\left[\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)}\right] \\
&= \frac{sY(t) - \delta K(t)}{A(t)L(t)} - k(t)(n+g) \\
&= sf(k(t)) - (\delta + n + g)k(t)
\end{aligned}$$

The first term, $sf(k(t))$, is the actual investment in physical capital per unit of effective labor. The second term, $(\delta + n + g)k(t)$, is the *effective* depreciation of capital per unit of effective labor.

Steady-state (equilibrium) occurs at such value of capital per effective labor, k^*, where $\dot{k}(t) = 0$:

$$sf(k^*) - (\delta + n + g)k^* = 0.$$

At k^*, investment equals effective depreciation and k remains constant over time. Because of the Inada conditions, there is only a single value of $k^* > 0$.

Behavior of aggregate variables in the steady state: effective labor, AL, grows at rate $(n+g)$. Capital grows at the same rate (note that $K = ALk^*$, with k^* constant). Because of CRTS, aggregate output grows at rate $(n+g)$.

Output per unit of effective labor, y, is constant. Capital per worker, $\frac{K}{L}$, and output per worker, $\frac{Y}{L}$, grow at rate g.

Balanced growth path

In the steady state, all variables grow at constant rates:

- Capital per unit of effective labor, k^*: constant;
- Labor and technology grow at rates n and g, respectively;
- Capital, $K = ALk$ grows at rate $(n+g)$;

- Because of CRTS, output, Y, also grows at rate $(n + g)$;
- Capital per worker, $\frac{K}{L}$, and output per worker, $\frac{Y}{L}$, grow at rate g.

Hence, the equilibrium (steady state) rate of growth of output per capita is determined by the rate of technological progress only.

Comparative Statics: Change in the Savings Rate

Recall: in the steady state:

$$sf(k^*) = (\delta + n + g) k^*$$

The savings rate, s, is a key parameter of the Solow model. An increase in s implies higher actual investment; k grows until it reaches its new (higher) steady-state value. In the transition to the new steady state, the rate of growth of output per worker accelerates.

Once the new steady state is attained, all variables grow again at the same rates as before; output per worker again grows at the rate of growth of technological progress, g, which is independent of s. An increase in the savings rate only leads to a temporary increase in the growth rate of output per worker (but a permanent rise in the level of capital per worker and output per worker).

In the Solow model, only changes in technological progress have permanent *growth effects*, all other changes have *level effects* only.

Effect on Consumption

Household welfare depends on consumption rather than output.

Fraction of output that is consumed is $(1-s)$ so that $c = (1-s) f(k)$. When s increases, c initially falls but then rises gradually as output per worker, $f(k)$, increases. Eventually, it may be greater or smaller than before the change in s.

In the steady state (using $sf(k^*) = (\delta + n + g) k^*$)

$$\begin{aligned} c^* &= f(k^*) - sf(k^*) \\ &= f(k^*) - (\delta + n + g) k^* \end{aligned}$$

How do changes in the savings rate affect consumption? Noting that k^* depends on s as well as n, g, and δ:

$$\frac{\partial c^*}{\partial s} = \left[f\prime (k^*(s,n,g,\delta)) - (\delta + n + g) \right] \frac{\partial k^*(s,n,g,\delta)}{\partial s}.$$

The last term, $\frac{\partial k^*(s,n,g,\delta)}{\partial s}$, is always positive: an increase in the savings rate always translates into a higher k^*.

The impact of changes in the savings rate on consumption depends on the sign of $[f\prime(k^*) - (\delta + n + g)]$, i.e. whether the marginal product of capital, $f\prime(k^*)$, exceeds $(\delta + n + g)$. Because of the properties that we assumed to hold for the production function, $f\prime(k^*)$ exceeds $(\delta + n + g)$ for small values of k and falls short of it for large values of k.

Golden-rule level of capital: The highest possible level of consumption is attained at the level of k^* such that

$$\frac{\partial c^*}{\partial s} = 0$$

or

$$f\prime(k^*) = (\delta + n + g)$$

Note: in the Solow model, households/firms do not make any optimization

decisions; the savings rate is exogenous. Therefore, there is no reason to expect the golden-rule level of k^* to prevail rather than some other level of k^*.

Effect on Output

Effect on steady-state output, y^*:

$$\frac{\partial y^*}{\partial s} = f\prime(k^*) \frac{\partial k^*(s,n,g,\delta)}{\partial s}$$

In the steady state $\dot{k} = 0$, therefore we can use the following property

$$sf(k^*(s,n,g,\delta)) = (\delta + n + g) k^*(s,n,g,\delta).$$

Differentiating both sides w.r.t. s:

$$\begin{aligned} sf\prime(k^*) \frac{\partial k^*}{\partial s} + f(k^*) &= (\delta + n + g) \frac{\partial k^*}{\partial s} \\ f(k^*) &= [(\delta + n + g) - sf\prime(k^*)] \frac{\partial k^*}{\partial s} \end{aligned}$$

We can solve for $\frac{\partial k^*}{\partial s}$:

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(\delta + n + g) - sf\prime(k^*)}.$$

and substitute this back into the expression for $\frac{\partial y^*}{\partial s}$ above. The effect of changes in the savings rate on output then is

$$\frac{\partial y^*}{\partial s} = f\prime(k^*) \frac{f(k^*)}{(\delta + n + g) - sf\prime(k^*)}.$$

Elasticity of output w.r.t. the savings rate can be obtained by multiplying both sides of the above equation by $\frac{s}{y^*}$:

$$\frac{\partial y^*}{\partial s} \frac{s}{y^*} = \frac{s}{f(k^*)} \frac{f\prime(k^*) f(k^*)}{(\delta + n + g) - sf\prime(k^*)}.$$

Note that in the steady state, $sf(k^*) = (\delta + n + g) k^*$ and $s = (\delta + n + g) k^*/f(k^*)$:

$$\begin{aligned} \frac{\partial y^*}{\partial s} \frac{s}{y^*} &= \frac{(\delta + n + g) k^* f\prime(k^*)}{f(k^*) [(\delta + n + g) - (\delta + n + g) k^* f\prime(k^*)/f(k^*)]} \\ &= \frac{k^* f\prime(k^*)}{f(k^*)[1 - k^* f\prime(k^*)/f(k^*)]} \\ &= \frac{k^* f\prime(k^*)/f(k^*)}{1 - k^* f\prime(k^*)/f(k^*)} \end{aligned}$$

$k^* f\prime(k^*)/f(k^*)$ is the elasticity of output w.r.t. capital at $k = k^*$. Denoting this $\alpha_K(k^*)$:

$$\frac{\partial y^*}{\partial s}\frac{s}{y^*} = \frac{\alpha_K(k^*)}{1-\alpha_K(k^*)}.$$

To evaluate this, note that under competitive markers and in absence of externalities, capital earns its marginal product

$$\frac{\partial Y}{\partial K} = \frac{\partial [ALf(k)]}{\partial k}\frac{\partial k}{\partial K} = ALf\prime(k)\frac{1}{AL} = f\prime(k).$$

The share of output earned by capital in the steady state is

$$\frac{k^* f\prime(k^*)}{y} = \alpha_K(k^*).$$

Given that this is approximately $\frac{1}{3}$ in most countries, $\frac{\partial y^*}{\partial s}\frac{s}{y^*} \approx \frac{1}{2}$. Thus, 1% increase in the savings rate increases the steady-state level of output per effective labor by approximately 0.5% – a modest effect.

Relevance of the Solow Model

Two sources of variation in output per worker according to the Solow model:

- Differences in capital per worker, $\frac{K}{L}$ (these, in turn, depend on differences in the savings rate and population growth);
- Differences in knowledge, A.

The notion of *knowledge* is very vague in the Solow model: the very variable that drives growth in the steady state is not analyzed by the model at all!

Differences in capital per worker can only account for a fraction of the observed differences over time or across countries. Assuming a Cobb-Douglas production function, $y = k^{\alpha}$ with $\alpha = \frac{1}{3}$, a 10-fold difference in output per worker would require a 1000-fold difference in capital per worker:

$$\begin{aligned} \frac{Y}{L} &= Ak^{\frac{1}{3}} \\ 10\frac{Y}{L} &= A\left(1000k\right)^{\frac{1}{3}} \end{aligned}$$

Given the differences in capital per worker between rich and poor countries, the marginal return to capital should be much higher in the latter and we should observe large flows of capital from rich to poor countries. Neither is the case, however (Lucas, 1990, AER Papers and Proceedings).

Assume all countries share the same CRTS production function: $Y = F(K, AL)$, and have access to the same technology. If we observe differences in output per worker, these must be due to differences in capital per worker. Then, the marginal product of capital (and the return to capital) should be correspondingly higher in the poorer country.

If movement of capital across countries is unrestricted, capital should flow from rich to poor countries until capital per worker is equalized across countries (so that returns to capital and to labor are equalized).

Assume standard Cobb-Douglas production function, $Y = K^{\alpha}(AL)^{1-\alpha}$, or $y = k^{\alpha}$ in intensive form (NB this implies $y^{\frac{1}{\alpha}} = k$). Then, the marginal product of capital is

$$\begin{aligned} f'(k) &= \alpha k^{\alpha-1} \\ &= \alpha y^{\frac{\alpha-1}{\alpha}}. \end{aligned}$$

Consider two countries, US and India, and assume that output per worker is 10 times higher in the US than in India. Then, the marginal product of capital in India is

$$\begin{aligned}
[f\prime(k)]_{IN} &= \alpha\,(y_{IN})^{\frac{\alpha-1}{\alpha}} \\
&= \alpha\left(\frac{y_{US}}{10}\right)^{\frac{\alpha-1}{\alpha}} \\
&= \left(\frac{1}{10}\right)^{\frac{\alpha-1}{\alpha}} \alpha\,(y_{US})^{\frac{\alpha-1}{\alpha}} \\
&= 10^{\frac{1-\alpha}{\alpha}}\,[f\prime(k)]_{US}\,.
\end{aligned}$$

For $\alpha = \frac{1}{3}$, the return to capital in India should be 100 higher than the return to capital in the US!

Lucas (1990) observes that actual differences in interest rates are nowhere near those predicted by the standard growth theory. He offers the following candidate explanations:

1. *Differences in labor productivity:* Let's assume US workers are 5 times more productive than Indian workers:

   $$y_{IN} = \frac{Y_{IN}}{AL_{IN}} = \frac{1}{10}\frac{Y_{US}}{AL_{US}/5} = \frac{1}{2}y_{US}$$

   The preceding derivation then implies that the return to capital should be 4 times higher in India than in the US so that the incentive for movement of capital from rich to poor countries persists;

2. *Absence of knowledge spillovers across countries*: countries do not have access to the same production technology;

3. *Political risk*: threat of expropriation of profits by the poor-country government;

4. *Restrictions on capital inflows* in order to take advantage of monopoly position (initially by the colonial government, then the poor country government): restricting capital inflows translates into higher return on capital and low wages –> monopoly profit is maximized.

In the absence of barriers to capital mobility, an increase in the savings rate in one country should translate into higher investment in which ever country has the highest greturn to capital. If the return to capital is equalized across countries, an increase in the savings rate in one country should be spread uniformly across all countries.

Feldstein and Horioka (EJ, 1980) measure the correlation between investment and savings rates across 21 industrialized countries between 1960 and 1974 and find them to be very strongly (almost perfectly) correlated.

Possible explanations:

1. Barriers to capital mobility;
2. Some variables may affect both savings and investment (e.g. tax policy);
3. Governments may implement policies that increase the correlation of savings and investment in order to avoid large trade deficits/surpluses.

The Feldstein-Horioka result is even more puzzling because it is obtained in a cross section of developed industrialized countries for which differences in labor productivity, knowledge spillovers or political risk are small. Hence, restrictions on capital flows remain important, even among developed, countries.

Recent literature increasingly looks at differences in the quality of the institutional environment.

Solow Model and Convergence

Solow model predicts convergence:

1. Countries are predicted to converge to their respective steady states (*conditional convergence*).
2. The rate of return is lower in countries with higher capital per worker, therefore capital should flow to poorer countries, thus driving convergence.
3. Diffusion of knowledge: poorer countries can adopt modern technologies without having to develop them and thus benefit from technological progress in more advanced countries.

Empirical tests: regress growth on a measure of initial income:

- Negative coefficient –> convergence;
- Coefficient of 0: no convergence.

Results: mixed at best. Two problems:

1. *Selection bias:* Long data series are mainly available for industrialized countries –> countries that used to be poor in the past are more likely to have long data series if they are rich today.
2. *Measurement error:* Historical data are more likely to be imprecisely measured than current data. Growth appears high if initial income is underestimated and low if initial income is overestimated, hence it is impossible to distinguish convergence from measurement error.

De Long (1988): accounting for plausible magnitute of the measurement error (variance of the estimated initial income around its true value) yields absence of convergence or even divergence.

Emprical Evidence on Growth

Barro (QJE 1991): only loosely motivated by growth theory (i.e. regressions estimated without deriving explicit predictions based on theory), relates growth to a number of variables that are expected to be important for growth performance.

Determinants of growth: initial income (-); savings and human capital (+) ; government consumption (-); political instability and price distortions (-); dummies for Sub-saharan Africa and Latin America (-).

Determinants of fertility (population growth): initial income (-); human capital (-).

Determinants of the investment share in GDP: initial income (-) [convergence in capital stock]; human capital (+) [interaction between investment in physical and human capital]; political instability and price distortions (-).

Levine and Renelt (AER 1992): test of robustness of numerous *candidate* variables (used elsewhere in the literature).

Regressions always include four *basic* variables motivated by the theory: investment share, initial per-capita GDP, secondary-school enrollment, and population growth.

Candidate variables are included alongside the basic variables and various combinations of additional variables to test their robustness.

Extreme bounds analysis: If the candidate variable is always significant and of the expected sign, it is accepted as *robust*.

Results: very few robust variables: investment share, initial income, secondary school enrollment, trade distortions, revolutions and coups.

Sala-i-Martin (AER Papers and Proceedings, 1997): EBA too extreme and tends to reject too many variables. Instead, Sala-i-Martin considers the full distribution of regression coefficients and accepts them as robust if 95% of the distribution is above (below) zero.

Three *basic* variables are always included: initial income, initial life expectancy, and primary school enrollment. 59 *candidate* variables are included alongside the basic variables and all possible combinations of 3 of the remaining candidate variables: in total, 2 million regressions.

22 variables are found to be robust: Geography (Sub-Saharan Africa, Latin America, absolute latitude); Political institutions (rule of law, political rights and civil liberties, revolutions and coups, wars); Religion; Market distortions; Investment in physical capital; Primary-sector production; Openness; Degree of capitalism; and History (former Spanish colonies).

Should We Take Solow Seriously?

Mankiw, Romer and Weil (QJE, 1992) derive testable predictions of the Solow growth model and put them to an empirical test.

Cobb-Douglas production function:

$$Y(t) = K(t)^{\alpha} (A(t) L(t))^{1-\alpha}$$

with $0 < \alpha < 1$. Labor and knowledge are assumed to grow at exogenous rates n and g:

$$L(t) = L(0) e^{nt}$$
$$A(t) = A(0) e^{gt}$$

The savings rate is constant, s. Depreciation rate of physical capital is δ. Denoting $k = \frac{K}{AL}$, we get

$$\begin{aligned} \dot{k}(t) &= sy(t) - (\delta + n + g) k(t) \\ &= sk(t)^{\alpha} - (\delta + n + g) k(t). \end{aligned}$$

The steady-state value of capital per unit of effective labor then is

$$k^* = \left(\frac{s}{\delta + n + g} \right)^{\frac{1}{1-\alpha}}.$$

The steady-state level of capital responds positively to the savings rate and negatively to population growth.

Substituting steady-state capital into the production function, we get an expression for steady-state output per effective labor and output per worker

$$y = \left(\frac{s}{\delta + n + g} \right)^{\frac{\alpha}{1-\alpha}}$$

$$\frac{Y(t)}{L(t)} = A \left(\frac{s}{\delta + n + g} \right)^{\frac{\alpha}{1-\alpha}}.$$

Substituting for knowledge, $A(t) = A(0) e^{gt}$, and rewriting in logs:

$$\ln \frac{Y(t)}{L(t)} = A(0) + gt + \frac{\alpha}{1-\alpha} \ln(s) - \frac{\alpha}{1-\alpha} \ln(\delta + n + g).$$

Assuming $A(0) = a + \epsilon$, this can be estimated as

$$\ln \frac{Y(t)}{L(t)} = a + gt + \frac{\alpha}{1-\alpha} \ln(s) - \frac{\alpha}{1-\alpha} \ln(\delta + n + g) + \epsilon.$$

The Solow model predicts not only the signs but also the magnitudes of the effects of the savings rate and population growth on output per worker. For $\alpha = \frac{1}{3}$, the elasticity of output per worker w.r.t. the savings rate should be 0.5 while that w.r.t. $(\delta + n + g)$ should be -0.5.

MRW estimate this equation, using Summers and Heston (1988) PWT data over 1960-1985. n is measured as the average growth rate of working-age (15-64) population. $\delta + g$ is assumed to be constant across countries and is estimated to be 0.05. s is measured as the average share of investment, including public investment, in GDP. $\frac{Y(t)}{L(t)}$ is measured in real terms as of 1985. The basic sample includes 98 countries (major oil producers excluded). The analysis is also carried out for an intermediate sample of 75 countries – excluding those with unreliable data and very small countries, and for 22 OECD countries.

Results: the model yields correct signs for the effects of the savings rate and population growth, they are approximately equal in magnitute, but their sizes are much larger than those predicted by the model. Nonetheless, the standard textbook Solow model explains more than half of the variance in the data on growth.

Augmented Solow model:

Add human capital to the production function

$$Y(t) = K(t)^{\alpha} H(t)^{\beta} (A(t) L(t))^{1-\alpha-\beta}$$

assuming $\alpha + \beta < 1$. Denoting investment in physical and human capital as s_k and s_h, $y = \frac{Y}{AL}$, $k = \frac{K}{AL}$ and $h = \frac{H}{AL}$, we get

$$\dot{k}(t) = s_k y(t) - (\delta + n + g) k(t)$$

$$\dot{h}(t) = s_h y(t) - (\delta + n + g) h(t)$$

This assumes that human capital is accumulated in the same way as physical capital and depreciates at the same rate. The production function in intensive form is

$$y(t) = k(t)^{\alpha} h(t)^{\beta}$$

The steady-state is defined by

$$k^* = \left(\frac{s_k^{1-\beta} s_h^{\beta}}{\delta + n + g} \right)^{\frac{1}{1-\alpha-\beta}}$$

$$h^* = \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{\delta + n + g} \right)^{\frac{1}{1-\alpha-\beta}}$$

and

$$
\begin{aligned}
y^* &= \left(\frac{s_k^{1-\beta} s_h^{\beta}}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha-\beta}} \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{\delta+n+g}\right)^{\frac{\beta}{1-\alpha-\beta}} \\
&= s_k^{\frac{\alpha}{1-\alpha-\beta}} s_h^{\frac{\beta}{1-\alpha-\beta}} \left(\delta+n+g\right)^{-\frac{\alpha+\beta}{1-\alpha-\beta}}
\end{aligned}
$$

In logs

$$
\ln\frac{Y(t)}{L(t)} = A(0) + gt - \frac{\alpha+\beta}{1-\alpha-\beta}\ln(\delta+n+g) + \frac{\alpha}{1-\alpha-\beta}\ln(s_k) + \frac{\beta}{1-\alpha-\beta}\ln(s_h)
$$

The augmented model predicts higher magnitudes for the effect of investment in physical capital: if $\alpha = \beta = \frac{1}{3}$, the elasticity is 1 rather than 0.5. The effect of $(\delta + n + g)$ is predicted to be larger in magnitude than that of the savings rate: -2.

Using the expression for steady-state human capital, we can rewrite this as

$$
\ln\frac{Y(t)}{L(t)} = A(0) + gt + \frac{\alpha}{1-\alpha}\ln(s_k) - \frac{\alpha}{1-\alpha}\ln(\delta+n+g) + \frac{\beta}{1-\alpha}\ln(h^*)
$$

This equation is almost identical to the one derived from the textbook Solow model. However, since h^* depends on both s_k and n, omitting human capital would result in biased coeficient estimates on these variables.

Either equation can be estimated – depending on whether data on human capital correspond to levels or accumulation. MRW use the share of working-age population that is in secondary school as a proxy for accumulation.

Results: human capital accumulation has a positive impact on output per worker and is significant. The coefficient on investment in physical capital falls. The coefficient on all three terms sum to zero. The model predicts almost 80% of the variance in the data.

The Solow model also helps explain the pattern of convergence (or the absence thereof) across countries. The model does not predict *unconditional convergence*, only that countries converge to their respective steady states (which are in turn determined by the savings rate and population growth).

When out of steady state, the growth rate is determined by initial income as well as by the savings rate and population growth.

Result: when accounting for the differences in population growth and savings rate (and human capital), the data support *conditional convergence* at the rate of approximately 2% per year, which is similar to the convergence rate predicted by the augmented model.

However, Solow model predicts that investment rate should be uncorrelated with long-term (steady-state) growth: investment determines steady-state output per worker but not steady-state growth rate of output per worked. Bond, Leblebicioglu and Schiantarelli (IZA DP No. 1174) find that investment is correlated both with output per worker and long-term growth rate: increase in investment implies higher long-term growth. Their finding therefore suggest that (some) endogenous growth models fit the actual patterns of growth better than Solow model.

Extension: Land and Natural Resources

The amount of land and the stock of natural resources are fixed. Since both are used as inputs in the production process, their limited nature can limit growth. Consider the Cobb-Douglas production function augmented to include land and resources:

$$Y(t) = K(t)^{\alpha} R(t)^{\beta} T(t)^{\gamma} [A(t) L(t)]^{1-\alpha-\beta-\gamma}$$

where R stands for natural resources, T is land, and $\alpha, \beta, \gamma > 0$ and $\alpha+\beta+\gamma < 1$.

Dynamics of inputs of production:

$$\begin{aligned}
\dot{K}(t) &= sY(t) - \delta K(t) \\
\dot{L}(t) &= nL(t) \\
\dot{A}(t) &= gA(t) \\
\dot{T}(t) &= 0 \\
\dot{R}(t) &= -bR(t)
\end{aligned}$$

Hence, the amount of land is fixed, and the stock of natural resources declines at rate $b > 0$.

Balanced growth path: all variables are growing at constant rate. A, L, T and R grow at constant rates by assumption. Recall that $\dot{K}(t) = sY(t) - \delta K(t)$ so that capital grows at a constant rate if

$$\frac{\dot{K}(t)}{K(t)} = s\frac{Y(t)}{K(t)} - \delta$$

is constant. This implies that $\frac{Y}{K}$ must be constant, or that Y and K must grow at the same rate.

Taking logs of $Y(t)$:

$$\ln Y(t) = \alpha \ln K(t) + \beta \ln R(t) + \gamma \ln T(t) + (1-\alpha-\beta-\gamma)[\ln A(t) + \ln L(t)].$$

Differentiating w.r.t t, we get growth rates:

$$g_Y = \alpha g_K + \beta g_R + \gamma g_T + (1-\alpha-\beta-\gamma)(g_A + g_L)$$

or, substituting for growth rates of A, L, T and R :

$$g_Y = \alpha g_K - \beta b + (1-\alpha-\beta-\gamma)(n+g).$$

Since on balanced growth path $g_Y = g_K$, we can solve for g_Y

$$g_Y^{bgp} = \frac{(1-\alpha-\beta-\gamma)(n+g) - \beta b}{1-\alpha}.$$

The growth rate of output per worker is

$$\begin{aligned}g_{Y/L}^{bgp} &= \frac{(1-\alpha-\beta-\gamma)(n+g)-\beta b}{1-\alpha}-n \\ &= \frac{(1-\alpha-\beta-\gamma)g-\beta b-(\beta+\gamma)n}{1-\alpha}.\end{aligned}$$

This can be either positive or negative. Technological progress encourages growth of output per worker, but falling land and resources per worker limits growth. With sufficiently dynamic technological progress, output per workers grows at a positive rate despite the limited land and natural resources.

Suppose land and resources grew at the same rate as labor, n. Then, replicating the above derivation:

$$\begin{aligned}g_Y &= \alpha g_K+\beta g_R+\gamma g_T+(1-\alpha-\beta-\gamma)(g_A+g_L) \\ &= \alpha g_K+\beta n+\gamma n+(1-\alpha-\beta-\gamma)(n+g) \\ &= \alpha g_K+(1-\alpha)n+(1-\alpha-\beta-\gamma)g\end{aligned}$$

so that

$$\begin{aligned}\tilde{g}_Y^{bgp} &= \frac{(1-\alpha-\beta-\gamma)g}{1-\alpha}+n \\ \tilde{g}_{Y/L}^{bgp} &= \frac{(1-\alpha-\beta-\gamma)g}{1-\alpha}.\end{aligned}$$

Therefore, the limitation that land and natural resources impose on growth of output per worker is

$$\tilde{g}_{Y/L}^{bgp}-g_{Y/L}^{bgp}=\frac{\beta b+(\beta+\gamma)n}{1-\alpha}.$$

Thus, the greater are the shares of land and natural resources, the lower is growth. Nordhaus (1992) estimates that this *drag* accounts for 0.0024 (0.24%) of annual growth.