The Ramsey Model

Reading:

Romer, Chapter 2-A;

Developed by Ramsey (1928), later developed further by Cass (1965) and Koopmans (1965).
Similar to the Solow model: labor and knowledge grow at exogenous rates.
Important difference: capital stock is determined by optimization decisions of households and firms.

Firms

There are many identical firms, each with the same production function $Y = F(K, AL)$. The production function displays the same properties as before. Firms hire labor and capital in competitive markets. For simplicity, we assume there is no depreciation of capital ($\delta = 0$).

Households

H identical infinitely-lived households. The size of each household grows at rate n. Each household member supplies one unit of labor and rents its capital to firms.

Household maximizes its lifitime utility:

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt$$

where $u(C(t))$ is the *instantaneous utility* of each member of the household, $\frac{L(t)}{H}$ is the number of members of the household and ρ is the discount rate.

Assume *constant relative risk aversion (CRRA)* utility function:

$$u(C(t) = \frac{C(t)^{1-\theta}}{1-\theta}$$

where $\theta > 0$ and $\rho - n - (1-\theta)g > 0$.

Note:

$$u'(C) = C^{-\theta} > 0$$
$$u''(C) = -\theta C^{-\theta-1} < 0$$

so that the *coefficient of relative risk aversion* is θ:

$$-C\frac{u''(C)}{u'(C)} = -C\frac{-\theta C^{-\theta-1}}{C^{-\theta}} = \theta.$$

Behavior of Households and Firms

Firms are competitive and earn zero profits. Firms hire capital and labor and pay them their marginal products. The real interest rate then is:

$$r(t) = f'(k(t)).$$

The real wage is $W(t) = \frac{\partial F(K,AL)}{\partial L}$. Since $Y = ALf(k)$ and $k = K/AL$:

$$\begin{aligned} W(t) &= Af(k) - Akf'(k) \\ &= A[f(k(t)) - k(t)f'(k(t))]. \end{aligned}$$

The wage per unit of effective labor then is

$$w(t) = f(k(t)) - k(t)f'(k(t)).$$

Households' Budget Constraint

Households take r and w as given. The budget constrain stipulates that the present value of consumption cannot exceed the sum of initial wealth and present value of labor income.

Define

$$R(t) = \int_{\tau=0}^{t} r(\tau)d\tau$$

so that one unit of the good invested at time 0 is worth $e^{R(t)}$ units at time t; $R(t)$ captures the effect of continuously compounding interest over the period $[0,t]$. Similarly, one unit of output at some future time t is worth $e^{-R(t)}$ units at time 0.

Household budget constraint then is:

$$\int_{t=0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}dt \leq \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}W(t)\frac{L(t)}{H}dt$$

or

$$\frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}[W(t) - C(t)]\frac{L(t)}{H}dt \geq 0.$$

The integral can be rewritten as a limit:

$$\lim_{s\to\infty}\left[\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)}\left[W(t)\frac{L(t)}{H} - C(t)\frac{L(t)}{H}\right]dt\right] \geq 0.$$

Households capital stock at any time s is

$$\frac{K(s)}{H} = e^{R(s)}\frac{K(0)}{H} + \int_{t=0}^{s} e^{R(s)-R(t)}[W(t) - C(t)]\frac{L(t)}{H}dt,$$

that is, household wealth at any time equals to the interest-compounded value of its initial wealth and its savings (positive or negative). This can be rewritten as

$$\frac{K(s)}{H} = e^{R(s)}\left[\frac{K(0)}{H} + \int_{t=0}^{s} e^{-R(t)}[W(t) - C(t)]\frac{L(t)}{H}dt\right]$$

so that the household budget constrain becomes

$$\lim_{s\to\infty} e^{-R(s)}\frac{K(s)}{H} \geq 0.$$

This implies that households cannot follow a path of consumption and investment that would result in negative net present value of wealth (*no-Ponzi-game condition*).

Households' Maximization Problem

Households maximize their lifetime utility subject to the budget constraint. All households are identical, therefore all will choose the same path of consumption and investment.

Denote consumption per unit of effective labor $c(t)$ so that $C(t) = A(t)c(t)$ (each worker has A units of effective labor) and

$$\begin{aligned}\frac{C(t)^{1-\theta}}{1-\theta} &= \frac{[A(t)c(t)]^{1-\theta}}{1-\theta} \\ &= \frac{[A(0)e^{gt}]^{1-\theta}c(t)^{1-\theta}}{1-\theta} \\ &= A(0)^{1-\theta}e^{(1-\theta)gt}\frac{c(t)^{1-\theta}}{1-\theta}.\end{aligned}$$

Household life-time utility function becomes

$$\begin{aligned}U &= \int_{t=0}^{\infty} e^{-\rho t}\frac{C(t)^{1-\theta}}{1-\theta}\frac{L(t)}{H}dt \\ &= \int_{t=0}^{\infty} e^{-\rho t}A(0)^{1-\theta}e^{(1-\theta)gt}\frac{c(t)^{1-\theta}}{1-\theta}\frac{L(0)e^{nt}}{H}dt \\ &= A(0)^{1-\theta}\frac{L(0)}{H}\int_{t=0}^{\infty} e^{-\rho t+(1-\theta)gt+nt}\frac{c(t)^{1-\theta}}{1-\theta}dt \\ &= B\int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt\end{aligned}$$

where $B \equiv A(0)^{1-\theta}\frac{L(0)}{H}$ and $\beta \equiv \rho - n - (1-\theta)g$. Note that we assumed β to be positive.

The budget constraint,

$$\int_{t=0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}dt \le \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}W(t)\frac{L(t)}{H}dt$$

can be rewritten in terms of capital, consumption and wage per effective labor:

$$\int_{t=0}^{\infty} e^{-R(t)}c(t)\frac{A(t)L(t)}{H}dt \le k(0)\frac{A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}w(t)\frac{A(t)L(t)}{H}dt$$

$$\int_{t=0}^{\infty} e^{-R(t)}c(t)e^{(n+g)t}\frac{A(0)L(0)}{H}dt \le k(0)\frac{A(0)L(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}w(t)e^{(n+g)t}\frac{A(0)L(0)}{H}dt$$

$$\int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}c(t)dt \le k(0) + \int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}w(t)dt.$$

The limit version of the budget constraint can be also rewritten

$$\begin{aligned}\lim_{s\to\infty} e^{-R(s)}\frac{K(s)}{H} &\ge 0 \\ \lim_{s\to\infty} e^{-R(s)}e^{(n+g)s}k(s)\frac{A(0)L(0)}{H} &\ge 0 \\ \lim_{s\to\infty} e^{-R(s)}e^{(n+g)s}k(s) &\ge 0\end{aligned}$$

Households Behavior

Households choose the path of $c(t)$ that maximizes their lifetime utility subject to the budget constraint. Because additional consumption always increases utility, $u'(C) > 0$, the budget constraint will be met as equality.

The Langrangean:

$$B\int_{t=0}^{\infty} e^{-\beta t}\frac{c(t)^{1-\theta}}{1-\theta}dt + \lambda\left[k(0) + \int_{t=0}^{\infty} e^{-R(t)}w(t)e^{(n+g)t}dt - \int_{t=0}^{\infty} e^{-R(t)}e^{(n+g)t}c(t)dt\right]$$

The household chooses $c(t)$ at any point in time according to the following FOC:

$$Be^{-\beta t}c(t)^{-\theta} = \lambda e^{-R(t)}e^{(n+g)t}$$

and according to the budget constraint.

Taking logs of the FOC:

$$\ln B + -\beta t - \theta\ln c(t) = \ln\lambda - R(t) + (n+g)t$$

$$\ln B + -\beta t - \theta\ln c(t) = \ln\lambda - \int_{\tau=0}^{t} r(\tau)d\tau + (n+g)t.$$

Taking derivatives w.r.t. t

$$-\beta - \theta\frac{\dot{c}(t)}{c(t)} = -r(t) + n + g$$

or

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - n - g - \beta}{\theta}.$$

Substituting $\beta = \rho - n - (1-\theta)g$

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho - \theta g}{\theta}$$

$$= \frac{r(t) - \rho}{\theta} - g$$

Since $C(t) = A(t)c(t)$, consumption per worker grows at the rate of growth of $c(t)$ plus the rate of growth of knowledge, g:

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}.$$

Hence, consumption per worker grows if the interest rate exceeds the discount rate and falls otherwise. This equation is referred to as the *Euler equation*; it describes how $c(t)$ evolves for any given value of $c(0)$. The household chooses $c(0)$ so as to satisfy the budget constraint: the present value of lifetime consumption must equal the initial wealth plus the present value of savings.

The Dynamics of the Economy

Dynamics of c: The Euler equation can be rewritten using $r(t) = f'(k(t))$

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \rho - \theta g}{\theta}.$$

$\dot{c}(t) = 0$ when $f'(k(t)) = \rho + \theta g$. Denote the level of k for which this is the case k^*. Consumption is incresing for all $k < k^*$ and falling for $k > k^*$.

See Figure 2.1 in the book.

Dynamics of k: Recall that, as we derived for the Solow model, $\dot{k}(t) = sf(k(t)) - (\delta + n + g)k(t)$. Here, assuming no depreciation and allowing savings to vary:

$$\dot{k}(t) = f(k(t)) - c(t) - (n+g)k(t).$$

For any k, $\dot{k} = 0$ when

$$c = f(k) - (n+g)k.$$

Hence, k will remain constant if consumption equals the difference between output and break-even investment. The first term is increasing in k with diminishing returns, the second term is linear in k. Therefore, the level of consuption that keeps k constant is hump-shaped in k and peaks at such k for which $f'(k) = n + g$ (*golden-rule* level of k). If consumption is lower, k is increasing, and vice versa.

See Figure 2.2 in the book.

Steady State: The value of k^* is given by $f'(k^*) = \rho + \theta g$. The golden-rule k is given by $f'(k^{GR}) = n + g$. By assumption

$$\rho - n - (1 - \theta)g > 0$$

or

$$\rho + \theta g > n + g.$$

Therefore, $k^* < k^{GR}$.

The lines characterizing $\dot{c} = 0$ and $\dot{k} = 0$ can be combined in a *phase diagram*. For every $k(0) > 0$, there is a unique level of c that is consistent with the household's intertemporal optimization and will bring the economy to the steady state. The set of all such combinations of c and k is referred to as the *saddle path*.

See Figures 2.3 - 2.5 in the book.

Balanced Growth Path

Solow and Ramsey models display similar properties in equilibrium:
(1) Capital, output and consumption per unit of effective labor are constant. The savings rate, $\frac{y-c}{y}$, is also constant (because both y and c are constant).
(2) K, Y and total consumption growth at rate $n + g$.
(3) Capital per worker, output per worker and consumption per worker grow at rate g.

Hence, the basic prediction of the Solow model are reproduced also in the Ramsey model: in the steady state, the rate of growth of output per worker is determined entirely by technological progress.

Note, however, that the golden-rule level of k will not be attained in the Ramsey model. In the Solow model, the savings rate is exogenous and therefore any level of k, including the golden-rule one, can constitute a steady state. In the Ramsey model, the equilibrium is such that $k^* < k^{GR}$.

In the Ramsey model, the savings rate is the outcome of households' intertemporal optimization rather than being exogenous. Choosing k^{GR} would lead to higher c, but since households discount future consumption, this is not optimal.

Fall in the Discount Rate

Consider an economy that is on the balanced growth path. Suppose the discount rate, ρ, falls unexpectedly.

The discount rate only affects the equation for consumption,

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k) - \rho - \theta g}{\theta}.$$

The steady-state capital is given by $f'(k^*) = \rho + \theta g$. If ρ falls, this means that that the new k^* is higher than the original equilibrium. To bring the economy on the saddle path, c must initially fall and then rise gradually along the saddle path to the new equilibrium. Once the new equilibrium is attained, both k and c are higher than in the original equilibrium.

See Figure 2.6 in the book.

Adjustment after changes in the discount rate is similar to changes in the savings rate in the Solow model: growth accelerates during the transition to the new equilibrium; once the adjustment is completed, all variables grow at the same rates as before. However, the savings rate is not constant during the transition in the Ramsey model.

Government Expenditure and Consumption Smoothing

Consider impact of government expenditure – and especially of unanticipated changes in government expenditure – on the life-time path of consumption.

Assume government purchases and consumes $G(t)$ per unit of effective labor. This expenditure is financed by lump-sum tax also equal to $G(t)$: the budget is always balanced. Government expenditure does not affect utility from private consuption: government spending is pure waste, or it finances public goods that do not replace private consuption.

Dynamics of capital:

$$\dot{k}(t) = f(k(t)) - c(t) - G(t) - (n+g)k(t).$$

Graphically, the introduction of government spending implies that the $\dot{k} = 0$ curve shifts down; the $\dot{c} = 0$ curve remains unaffected.

Budget constraint:

$$\int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(t) dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} [w(t) - G(t)] dt.$$

The effect of unanticipated permanent increase in $G(t)$: Assume the economy is in equilibrium. $G(t)$ increases unexpectedly and the increase is perceived to be permanent. The $\dot{k} = 0$ curve shifts down while the $\dot{c} = 0$ curve remains unaffected. Consumption *jumps* immediately to the new equilibrium. The equilibrium k^* remains the same. Because

$$r(t) = f'(k(t))$$

the real interest rate does not change as a result of a permanent increase in $G(t)$.

See Figure 2.8 in the book.

The effect of unanticipated temporary increase in $G(t)$: The $\dot{k} = 0$ curve shifts down while the $\dot{c} = 0$ curve remains unaffected, but the new equilibrium is only seen as temporary.

Because the instantaneous utility is concave, households prefer to smooth consumption over time. Therefore, the fall in consumption will be only partial. After the initial adjustment, consumption will gradually rise and k will fall, so as to bring the economy onto the saddle path by the time government expenditure falls again. Once G returns to the original level, the economy converges along the saddle path back to the initial equilibrium.

The size of the initial fall in c depends on the expected length of the increase in G.

After the increase in G, k first falls and then increases again. Therefore, r first gradually increases and then gradually falls back to the original level.

See Figure 2.9 in the book.

Hence, the interest rate should only respond to temporary increases in G but not to permanent ones.

Barro (JME 1987) tests this proposition using interest rates and war-related military expenditure in the UK over the period 1729 to 1918, and finds evidence consistent with the theoretical prediction.